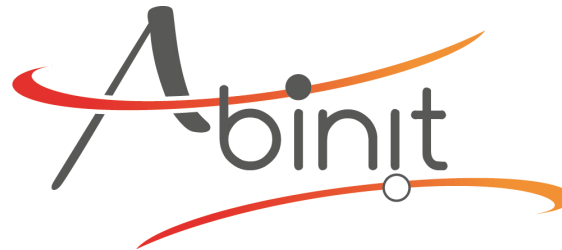


Implementation of DFT in



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Thanks to the > 50 ABINIT contributors, and especially to GM Rignanese for contributions to the slides

Density Functional Theory calculations

In ABINIT ...

Representation of mathematical formalism
with a **Plane Wave** basis set :

- wavefunctions
- density, potential

Periodic boundary conditions

=> wavefunctions characterized by a **wavevector** (k-vector)

PseudoPotentials (or Projector Augmented Waves – PAW)

Iterative techniques to solve the equations

(Schrödinger equation ; DFT Self-consistency ; optimisation
of atomic positions)

A basic 'input' file : dihydrogen (I)

H2 molecule in big cubic box
Characters after '#' or after '!' are comments, will be ignored.
Keywords followed by values. Order of keywords in file is not important.

Definition of the **unit cell**

acell 10 10 10 # Keyword "acell" refers to
 # lengths of primitive vectors (default in Bohr)

Definition of the **atom types**

ntypat 1 # Only one type of atom
znucl 1 # Keyword "znucl" refers to atomic number of
 # possible type(s) of atom. Pseudopotential(s)
 # mentioned in "filenames" file must correspond
 # to type(s) of atom. Here, the only type is Hydrogen.

Definition of the **atoms**

natom 2 # Two atoms
typat 1 1 # Both are of type 1, that is, Hydrogen
xcart # This keyword indicate that location of the atoms
 # will follow, one triplet of number for each atom
 # Triplet giving cartesian coordinates of atom 1, in Bohr
 # Triplet giving cartesian coordinates of atom 2, in Bohr

-0.7 0.0 0.0

0.7 0.0 0.0

A basic input file : dihydrogen (II)

Definition of **planewave basis set**

ecut 10.0 # Maximal plane-wave kinetic energy cut-off, in Hartree

Definition of **k-point grid**

kptopt 0 # Enter k points manually

nkpt 1 # Only one k point is needed for isolated system,
taken by default to be 0.0 0.0 0.0

#Definition of **SCF (self-consistent field) procedure**

nstep 10 # Maximal number of SCF cycles

toldfe 1.0d-6 # Will stop when, twice in a row, the difference
between two consecutive evaluations of total energy
differs by less than toldfe (default in Hartree)

diemac 2.0 # Although this is not mandatory, it is worth to precondition the
SCF cycle. A model dielectric function, used as standard
preconditioner, is described in "dielng" input variable section.
Here, we follow prescriptions for molecules in a big box

After modifying the following section, one might need to ...

###<BEGIN TEST_INFO> **Metadata ... to be ignored in the tutorial !**

Overview

1. Periodicity and specification of geometry in ABINIT
2. Plane wave basis set
3. Brillouin zone, wavevectors and occupation numbers
4. Computing the forces
5. Iterative algorithms (SCF, optimisation Kohn-Sham equation)
6. Computing band structure and density of states

References :

[Electronic Structure : Basic Theory and Practical Methods](#)

[Richard M. Martin](#)

[Cambridge University Press, 2004 \(ISBN: 0521782856\)](#)

[ABINIT project and ABINIT capabilities](#)

[X. Gonze et al, Comput. Mat. Science 25, 478 \(2002\)](#)

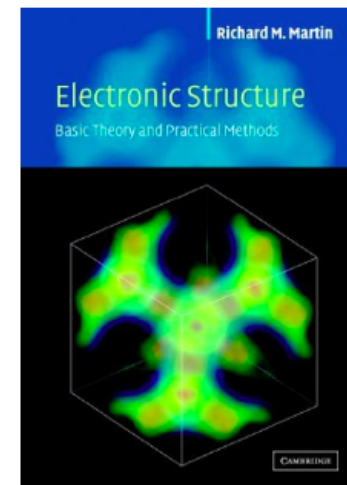
[X. Gonze et al, Comp. Phys. Comm. 180, 2582 \(2009\)](#)

[X. Gonze et al, Comp. Phys. Comm. 205, 106 \(2016\)](#)

[Focused papers on specific implementations within ABINIT \(e.g. PAW, parallelism\)](#)

[M. Torrent et al, Comput. Mat. Science 42, 337 \(2008\)](#)

[F. Bottin et al, Comp. Mat. Science 42, 329 \(2008\)](#)



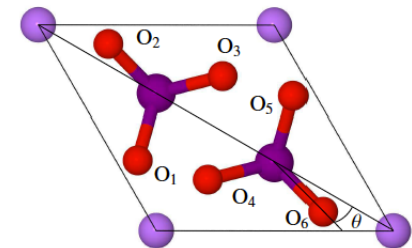
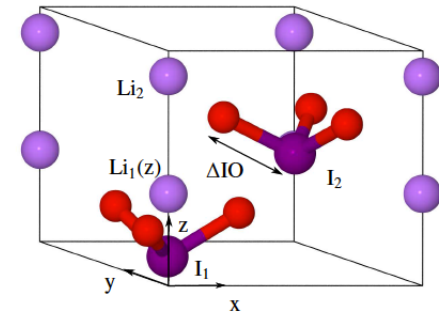
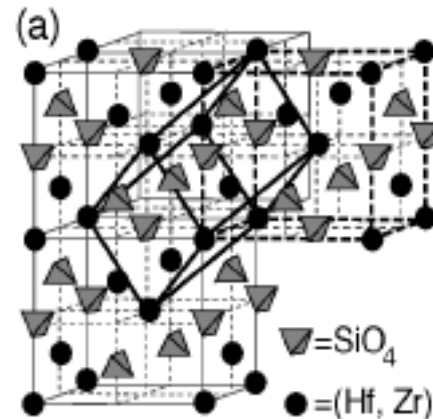
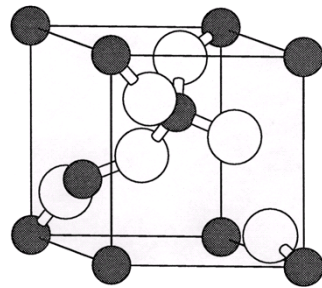
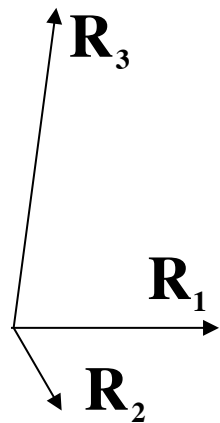
Periodicity and specification of the atomic geometry

ABINIT : a periodic code

Plane waves $e^{i\mathbf{K}\mathbf{r}}$: particularly simple and efficient (when used with pseudopotentials), but infinite spatial extent.

Cannot use a finite set of planewaves for finite systems !
Need periodic boundary conditions.

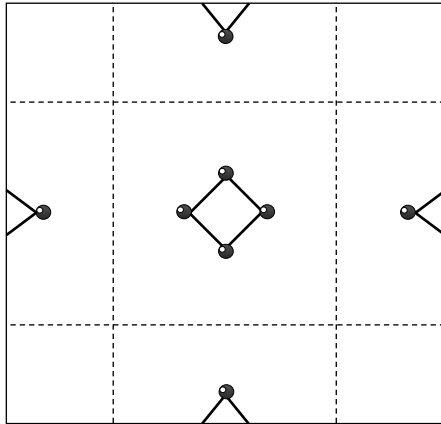
Primitive vectors \mathbf{R}_j , primitive cell volume Ω_0



OK for crystalline solids

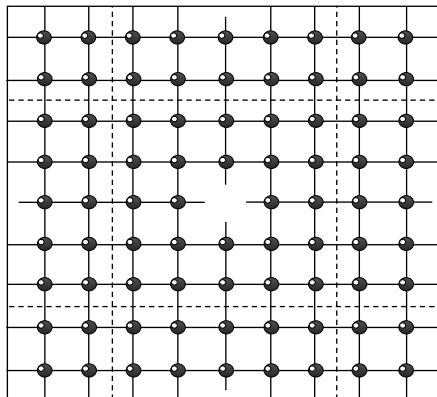
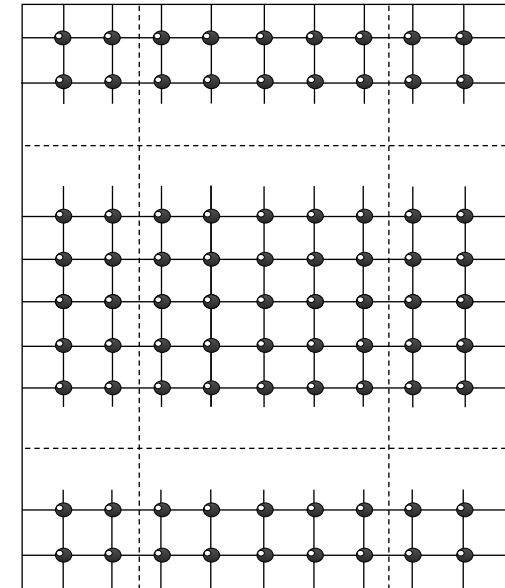
But : finite systems, surfaces, defects, polymers,
nanosystems ... ?

Solution : the supercell technique



Molecule,
cluster

Surface : treatment
of a slab
Interface

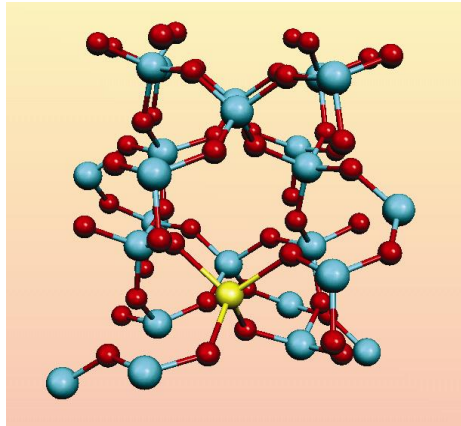


Point defect in a bulk solid

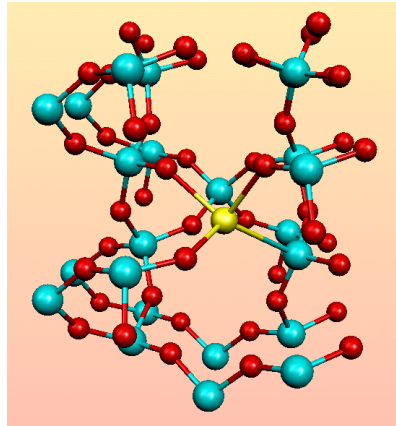
The supercell must be sufficiently big : convergence study

Examples of defects SiO_2 -quartz : Pb

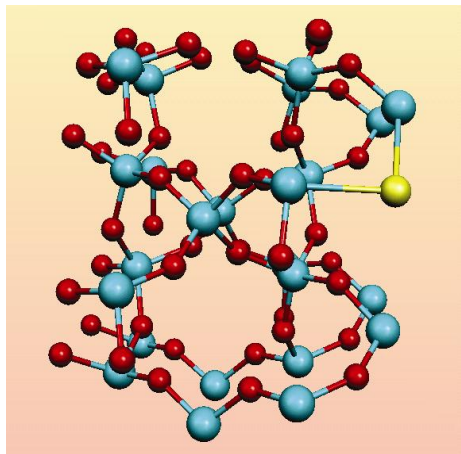
72-atom supercell of quartz



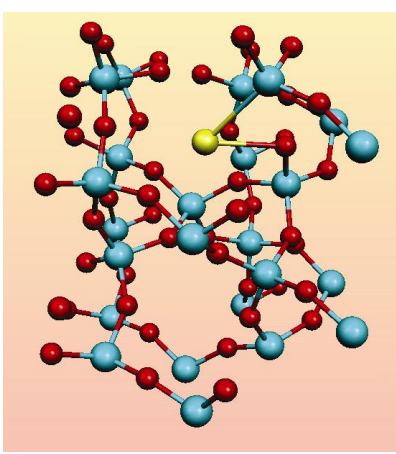
Pb^{Si}



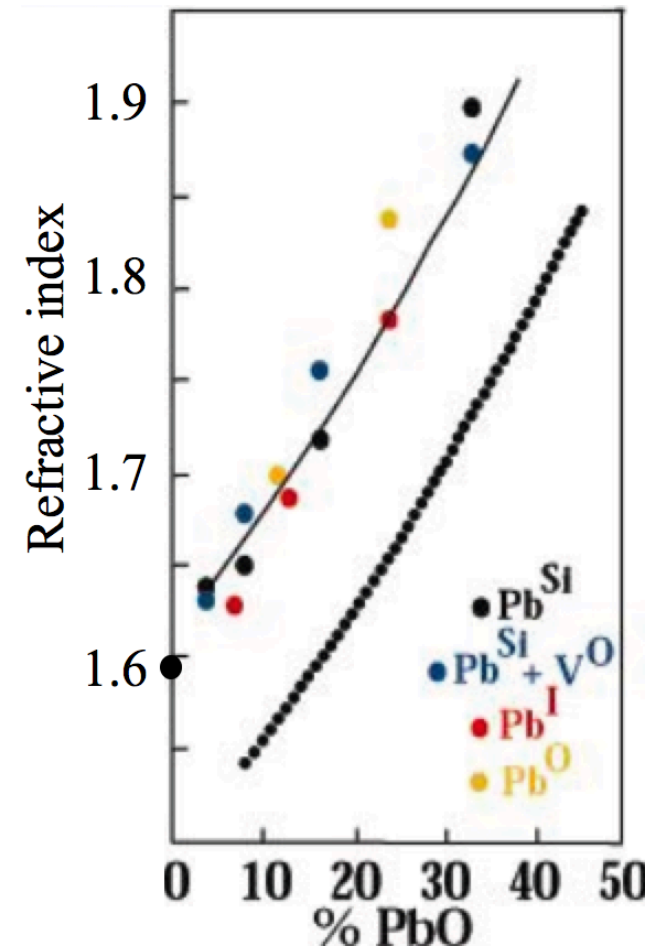
$\text{Pb}^{\text{Si}} + \text{V}^{\text{O}}$



Pb^{O}



Pb^{I}

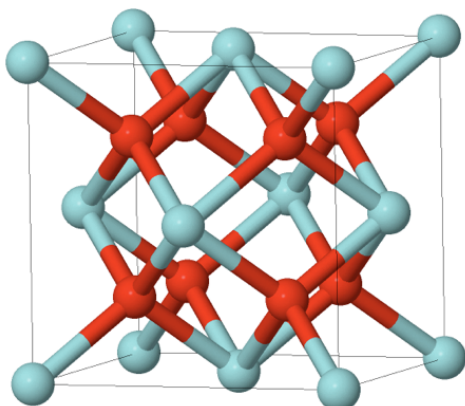


Comparison with amorphous SiO_2

Main input file : input variable flexibility

- cell primitive vectors → **rprim**
... or angle (degrees) between primitive vectors → **angdeg**
+ scale cell vector lengths → **acell**
+ scale cartesian coordinates → **scalecart**
- number of atoms → **natom**
- reduced coordinates → **xred** (initial guess ...)
... or cartesian coordinates → **xcart** (in Bohr) / **xangst** (in Å)
- type of atoms → **typat**
- space group → **spgroup** + **natrd**
... or number of symmetries → **nsym**
+ symmetry operations → **symrel** + **tnons**

Example : cubic zirconium dioxide



$Fm-3m$
 $a=5.010\text{\AA}$
 $b=5.010\text{\AA}$
 $c=5.010\text{\AA}$
 $\alpha=90.0^\circ$
 $\beta=90.0^\circ$
 $\gamma=90.0^\circ$

Bilbao Crystallographic Server → Assignment of Wyckoff Positions

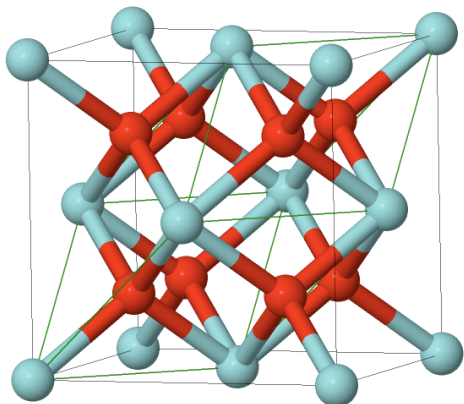
Assignment of Wyckoff Positions

Atoms Data:

AT.	WP	SS	Representative	Atomic orbit
Zr1	4a (0,0,0)	m-3m	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000) (0.000000, 0.500000, 0.500000) (0.500000, 0.000000, 0.500000) (0.500000, 0.500000, 0.000000)
O2	8c (1/4,1/4,1/4)	-43m	(0.250000, 0.250000, 0.250000)	(0.250000, 0.250000, 0.250000) (0.750000, 0.750000, 0.250000) (0.750000, 0.250000, 0.750000) (0.250000, 0.750000, 0.750000) (0.250000, 0.250000, 0.750000) (0.750000, 0.750000, 0.750000) (0.250000, 0.750000, 0.250000) (0.750000, 0.250000, 0.250000)

Face-centered cubic, with three atoms per primitive cell

Example : cubic zirconium dioxide



```
natom 3
acell 3*5.01 Angst
rprim 0.0 0.5 0.5
      0.5 0.0 0.5
      0.5 0.5 0.0
typat 1 2 2
xred 3*0.0 3*0.25 3*0.75
=> symmetries are found automatically
```

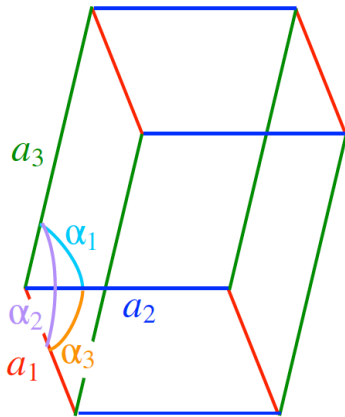
NOTE “*” is a repeater

OR

```
natom 3
acell 3*5.01 Angst
rprim 0.0 0.5 0.5
      0.5 0.0 0.5
      0.5 0.5 0.0
typat 1 2 2
spgroup 225 natrd 2
xred 3*0.0 3*0.25
=> the set of atoms is completed automatically
```

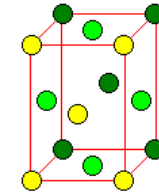
Primitive vectors in ABINIT (rprimd)

- $R_i(j) \rightarrow \text{rprimd}(j,i) = \text{scalecart}(j) \times \text{rprim}(j,i) \times \text{acell}(i)$



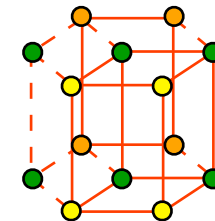
```
scalecart  9.5  9.8  10.0
rprim      0.0  0.5  0.5
           0.5  0.0  0.5
           0.5  0.5  0.0
```

face-centered
orthorhombic



```
acell      9.5  9.5  10.0
rprim      0.8660254038E+00  0.5  0.0
           -0.8660254038E+00  0.5  0.0
           0.0  0.0  1.0
```

hexagonal



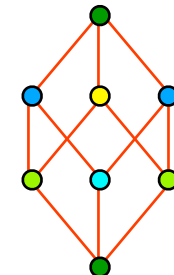
- $a_i \rightarrow \text{acell}(i) / \alpha_i \rightarrow \text{angdeg}(i)$

```
acell      9.5  9.5  10.0
angdeg     120  90  90
```

hexagonal

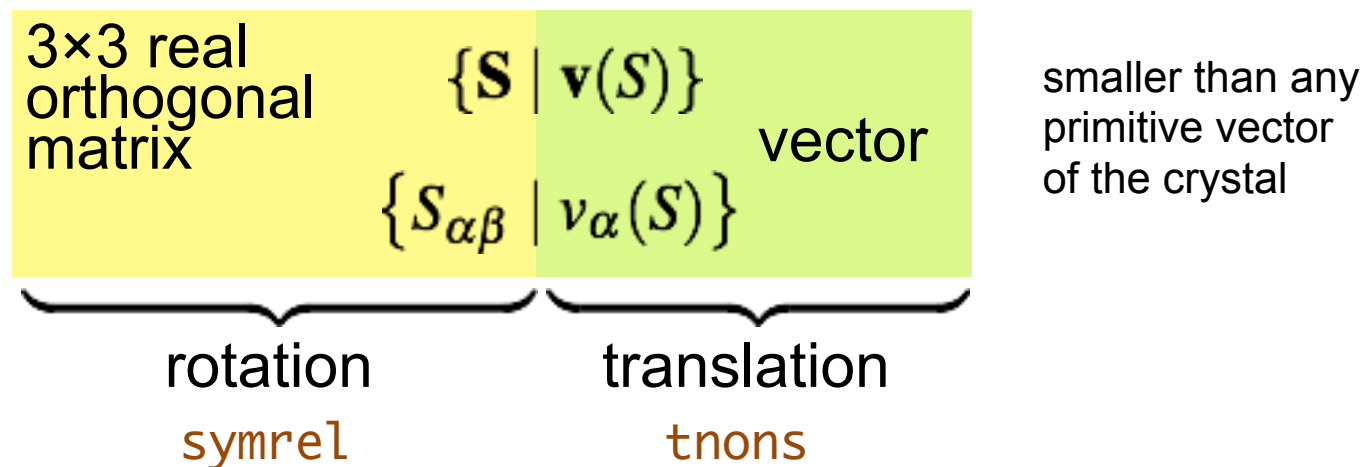
```
acell      9.0  9.0  9.0
angdeg     48  48  48
```

trigonal



Symmetries in ABINIT

- Seitz notation for the symmetry operations of crystal :



- Applied to the equilibrium position vector of atom κ relative to the origin of the cell τ_{κ} , this symmetry transforms it as:

$$\{\mathbf{S} \mid \mathbf{v}(S)\} \tau_{\kappa} = \mathbf{S} \tau_{\kappa} + \mathbf{v}(S) = \tau_{\kappa'} + \mathbf{R}^a$$

$$\{S_{\alpha\beta} \mid v_{\alpha}(S)\} \tau_{\kappa\alpha} = S_{\alpha\beta} \tau_{\kappa\alpha} + v_{\alpha}(S) = \tau_{\kappa'\alpha} + R_{\alpha}^a$$

where \mathbf{R}^a belongs to the real space lattice.

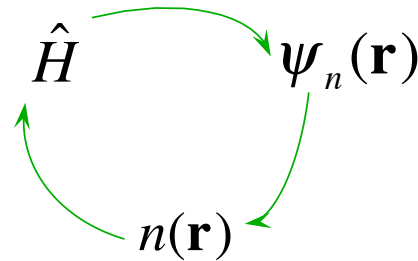
Plane wave basis set

$$\psi_{\mathbf{k}}(\mathbf{r}) = (N\Omega_0)^{-1/2} \sum_{\mathbf{G}} u_{\mathbf{k}}(\mathbf{G}) e^{i(\mathbf{k}+\mathbf{G})\mathbf{r}}$$

A reminder : basic equations in DFT

Solve self-consistently the Kohn-Sham equation

$$\left\{ \begin{array}{l} \hat{H} |\psi_n\rangle = \epsilon_n |\psi_n\rangle \\ \hat{H} = \hat{T} + \hat{V} + \hat{V}_{Hxc}[n] \\ n(\vec{r}) = \sum_n^{occ} \psi_n^*(\vec{r}) \psi_n(\vec{r}) \end{array} \right.$$



$$\delta_{mn} = \langle \psi_m | \psi_n \rangle \text{ for } m, n \in \text{occupied set}$$

or minimize

$$E_{el} \{ \psi \} = \sum_n^{occ} \langle \psi_n | \hat{T} + \hat{V} | \psi_n \rangle + E_{Hxc}[n]$$

with

$$\hat{V}(\vec{r}) = \sum_{a\kappa} - \frac{Z_\kappa}{|\vec{r} - \vec{R}_\kappa^a|}$$

Periodic system : wavevectors

For a **periodic** Hamiltonian : wavefunctions characterized by a wavevector \mathbf{k} (crystal momentum) in Brillouin Zone

Bloch theorem $\psi_{\mathbf{m},\mathbf{k}}(\mathbf{r}+\mathbf{R}_j) = e^{i\mathbf{k}\cdot\mathbf{R}_j} \psi_{\mathbf{m},\mathbf{k}}(\mathbf{r})$

$$\psi_{\mathbf{m},\mathbf{k}}(\mathbf{r}) = (N\Omega_0)^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{m},\mathbf{k}}(\mathbf{r}) \quad u_{\mathbf{m},\mathbf{k}}(\mathbf{r}+\mathbf{R}_j) = u_{\mathbf{m},\mathbf{k}}(\mathbf{r})$$

Normalization ?

Born-von Karman supercell supercell vectors $N_j\mathbf{R}_j$ with $N=N_1N_2N_3$

$$\psi_{\mathbf{m},\mathbf{k}}(\mathbf{r}+N_j\mathbf{R}_j) = \psi_{\mathbf{m},\mathbf{k}}(\mathbf{r})$$

Planewave basis set

Reciprocal lattice : set of \mathbf{G} vectors such that $e^{i\mathbf{G}\mathbf{r}_j} = 1$
 $e^{i\mathbf{G}\mathbf{r}}$ has the periodicity of the real lattice

$$u_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{G}} u_{\mathbf{k}}(\mathbf{G}) e^{i\mathbf{G}\mathbf{r}}$$

$$\psi_{\mathbf{k}}(\mathbf{r}) = (N\Omega_0)^{-1/2} \sum_{\mathbf{G}} u_{\mathbf{k}}(\mathbf{G}) e^{i(\mathbf{k}+\mathbf{G})\mathbf{r}}$$

$$u_{\mathbf{k}}(\mathbf{G}) = \frac{1}{\Omega_0} \int_{\Omega_0} e^{-i\mathbf{G}\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}) d\mathbf{r} \quad (\text{Fourier transform})$$

Kinetic energy of a plane wave $-\frac{\nabla^2}{2} \rightarrow \frac{(\mathbf{k}+\mathbf{G})^2}{2}$

The coefficients $u_{\mathbf{k}}(\mathbf{G})$ for the lowest eigenvectors decrease exponentially with the kinetic energy $\frac{(\mathbf{k} + \mathbf{G})^2}{2}$

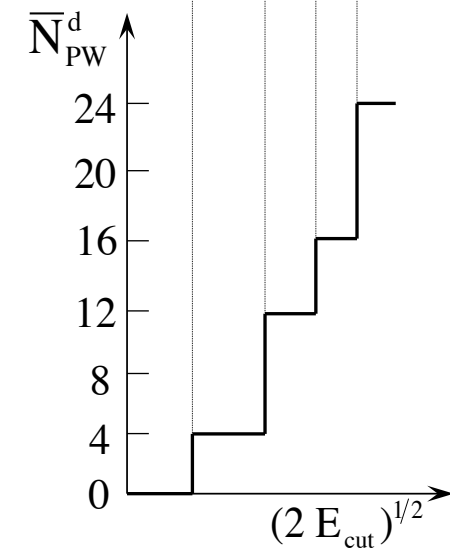
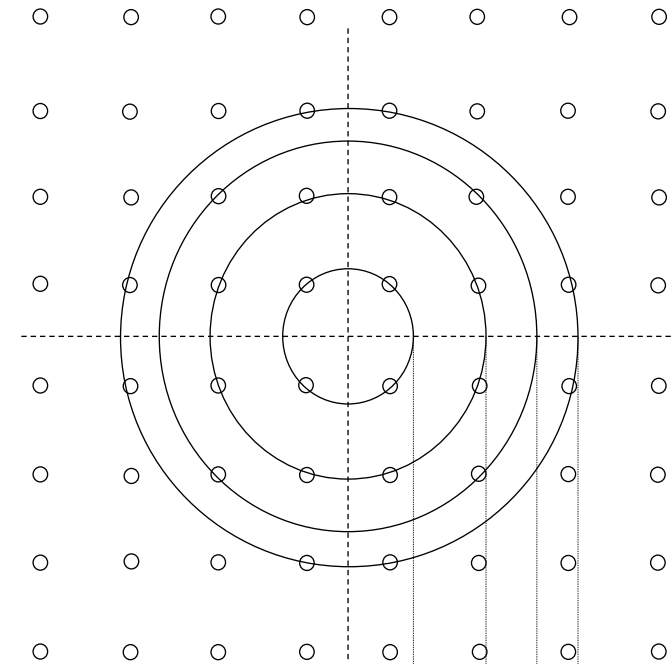
Selection of plane waves determined by a cut-off energy E_{cut}

$$\frac{(\mathbf{k} + \mathbf{G})^2}{2} < E_{\text{cut}} \quad \text{Plane wave sphere}$$

e_{cut}

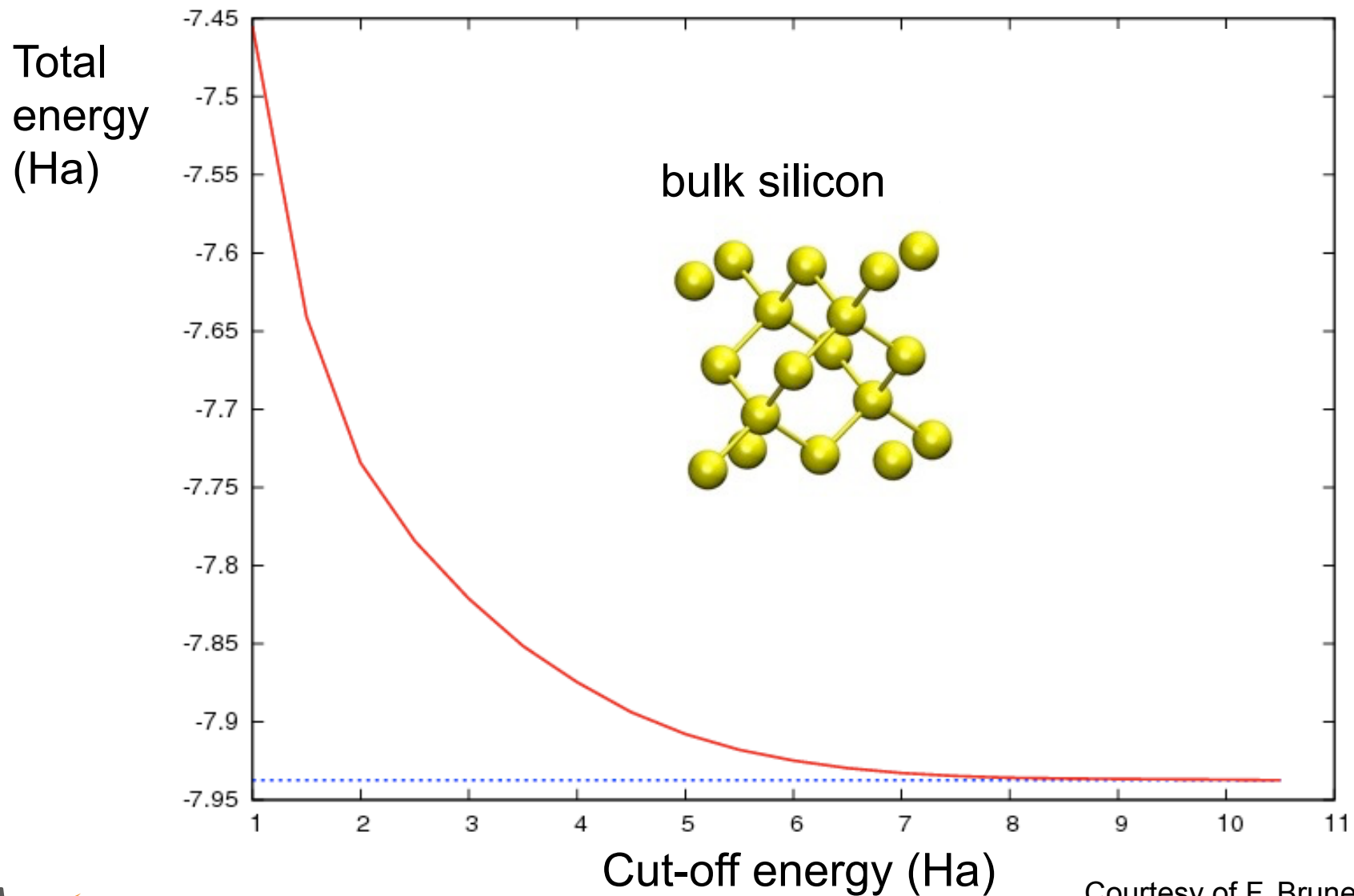
Number of planewaves

Number of plane waves
= function of the
kinetic energy cut-off
... **not continuous**



Also, a (discontinuous) function of
lattice parameter at fixed kinetic energy

Convergence wrt to kinetic energy cutoff



Courtesy of F. Bruneval

Plane waves : the density and potential

Fourier transform of a periodic function $f(\mathbf{r})$

$$f(\mathbf{G}) = \frac{1}{\Omega_{\text{or}}} \int_{\Omega_{\text{or}}} e^{-i\mathbf{G}\mathbf{r}} f(\mathbf{r}) d\mathbf{r} \quad f(\mathbf{r}) = \sum_{\bar{\mathbf{G}}} e^{i\mathbf{G}\mathbf{r}} f(\mathbf{G})$$

Poisson equation $\Rightarrow n(\mathbf{G})$ and $V_H(\mathbf{G})$

$$V_H(\mathbf{r}) = \int \frac{n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' \Leftrightarrow \nabla^2 V_H|_{\mathbf{r}} = -4\pi n(\mathbf{r})$$

Relation between Fourier coefficients:

$$G^2 V_H(\mathbf{G}) = 4\pi n(\mathbf{G}) \quad V_H(\mathbf{G}) = \frac{4\pi}{G^2} n(\mathbf{G})$$

For $G^2 = 0$ ($\mathbf{G} = 0$) divergence of V_H ($\mathbf{G} = 0$)

$$n(\mathbf{G} = 0) = \frac{1}{\Omega_{\text{or}}} \int_{\Omega_{\text{or}}} n(\mathbf{r}) d\mathbf{r} \quad \text{Average}$$

Representation of the density

Density associated with one eigenfunction :

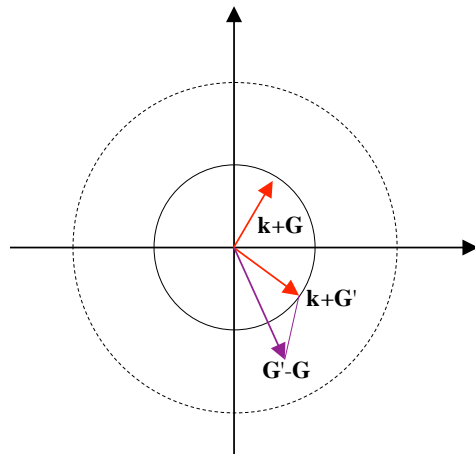
$$n_{\mathbf{n}\mathbf{k}}(\mathbf{r}) = u_{\mathbf{n}\mathbf{k}}^*(\mathbf{r}) u_{\mathbf{n}\mathbf{k}}(\mathbf{r})$$

Computation of

$$u_{\mathbf{n}\mathbf{k}}^*(\mathbf{r}) u_{\mathbf{n}\mathbf{k}}(\mathbf{r})$$

$$= \left(\sum_{\mathbf{G}} u_{\mathbf{n}\mathbf{k}}^*(\mathbf{G}) e^{i\mathbf{G}\mathbf{r}} \right) \left(\sum_{\mathbf{G}'} u_{\mathbf{n}\mathbf{k}}(\mathbf{G}') e^{-i\mathbf{G}'\mathbf{r}} \right)$$

$$= \sum_{\mathbf{G}\mathbf{G}'} \left[u_{\mathbf{n}\mathbf{k}}^*(\mathbf{G}) u_{\mathbf{n}\mathbf{k}}(\mathbf{G}') \right] e^{i(\mathbf{G}'-\mathbf{G})\mathbf{r}}$$



Non-zero coefficients for $\mathbf{k}+\mathbf{G} \in$ sphere

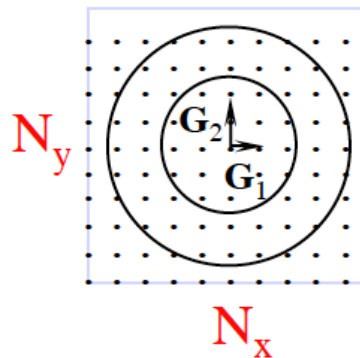
$\mathbf{k}+\mathbf{G}' \in$ sphere

The sphere for $n(\mathbf{G})$ has a double radius

From real space to reciprocal space

$$n(\mathbf{r}) = \sum_{\mathbf{G} \in \text{sphere}(2)} n(\mathbf{G}) e^{i\mathbf{G}\mathbf{r}}$$

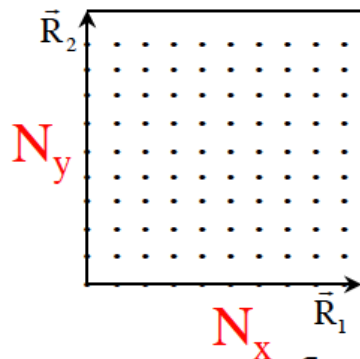
Use of the discrete Fourier transform $\{\mathbf{r}_i\} \leftrightarrow \{\mathbf{G}\}$



Reciprocal lattice

$$n(\mathbf{G}) = \frac{1}{N_{\mathbf{r}_i}} \sum_{\{\mathbf{r}_i\}} n(\mathbf{r}_i) e^{-i\mathbf{G}\mathbf{r}_i}$$

Fast Fourier Transform algorithm



Real lattice: original cell

Representation : wrap-up

- Choice of a basis (e.g. Plane waves)
- Truncating of the basis -> finite basis

$$\frac{(\mathbf{k}+\mathbf{G})^2}{2} < E_{\text{cut}} \quad \text{Sphere of plane waves}$$

- Discontinuous increase of the number of plane waves ?

Smearing of $u(\mathbf{G})$

-> Progressive incorporation of new \mathbf{G} vectors

- Representation of the density

Sphere with a double radius in the reciprocal space

- Going from the real space to reciprocal space

Discrete Fourier transform

Grid of points + Fast Fourier Transform

$$\{\mathbf{r}_i\} \leftrightarrow \{\mathbf{G}\}$$

Brillouin zone, wavevectors and occupation numbers

From discrete states to Brillouin zone

Discrete summations over states :

$$\text{Total kinetic energy} \quad \sum_n \langle \psi_n | -\frac{1}{2} \nabla^2 | \psi_n \rangle$$

$$\text{Density} \quad n(\mathbf{r}) = \sum_n \psi_n^*(\mathbf{r}) \psi_n(\mathbf{r})$$

In the periodic case : summation over energy bands +
integration over the Brillouin zone

$$\text{Total kinetic energy} \quad \sum_n \frac{1}{\Omega_{0k}} \int_{\Omega_{0k}} f(\epsilon_F - \epsilon_{nk}) \langle \psi_{nk} | -\frac{1}{2} \nabla^2 | \psi_{nk} \rangle dk$$

$$\text{Density} \quad n(\mathbf{r}) = \sum_n \frac{1}{\Omega_{0k}} \int_{\Omega_{0k}} f(\epsilon_F - \epsilon_{nk}) \psi_{nk}^*(\mathbf{r}) \psi_{nk}(\mathbf{r}) dk$$

How to treat $\frac{1}{\Omega_{0k}} \int_{\Omega_{0k}} X_k dk$?

Brillouin zone integration

$$\frac{1}{\Omega_{ok}} \int_{\Omega_{ok}} X_{\mathbf{k}} d\mathbf{k} \Rightarrow \sum_{\{\mathbf{k}\}} w_{\mathbf{k}} X_{\mathbf{k}} \quad [\text{with } \sum_{\{\mathbf{k}\}} w_{\mathbf{k}} = 1]$$

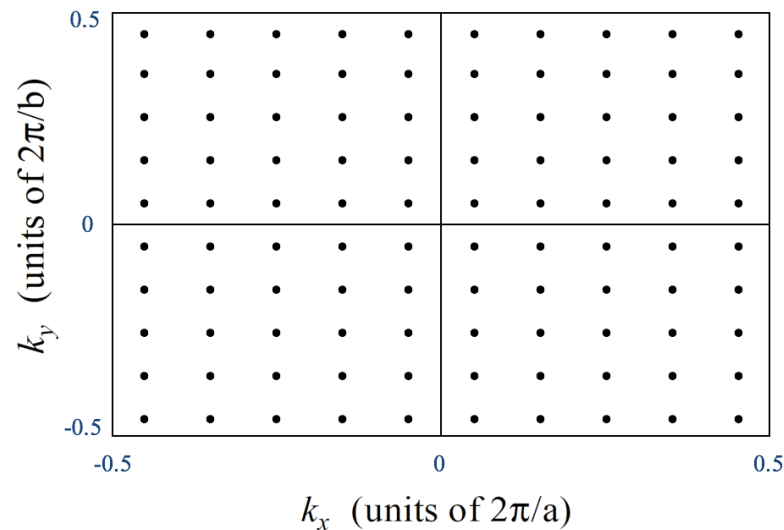
How to chose $\{\mathbf{k}\}$ and $\{w_{\mathbf{k}}\}$?

Special points

Weights

Simple answer : Homogeneous grid (1D - 2D - 3D) and equal weights

Homogeneous sampling of the Brillouin zone



Brillouin zone integration

Theorem :

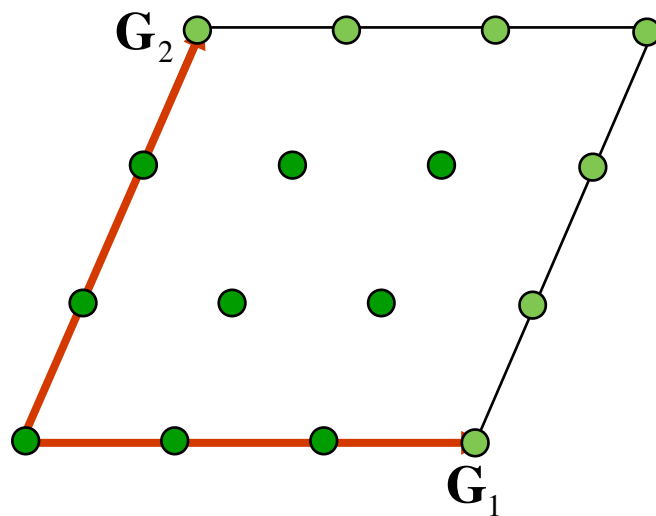
- If
- the integrand is periodic
 - the integrand is continuous + derivable at all orders $(C^\infty D^\infty)$
 - $\{\mathbf{k}\}$ homogeneous grid (1D - 2D - 3D) and $w_{\mathbf{k}}$ all equal

Then exponential convergence, with respect to $\Delta\mathbf{k}$

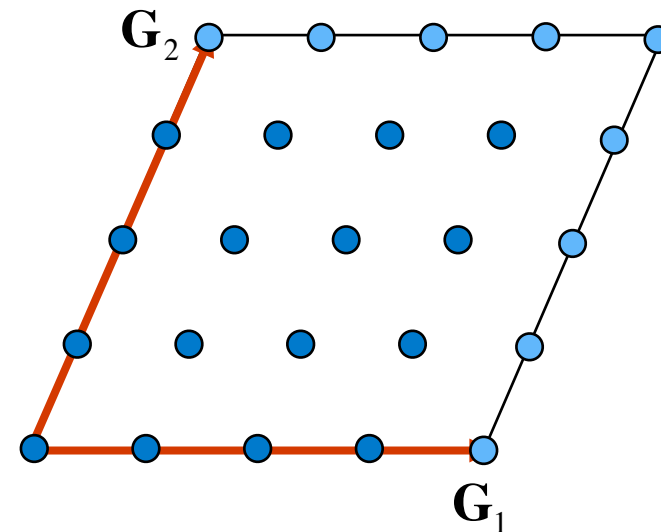
- OK for semiconductors/insulators where the occupation number is independent of \mathbf{k} within a band
- Convergence : one ought to test several grids with different $\Delta\mathbf{k}$
- Monkhorst & Pack grids (Phys. Rev. B 13, 5188 (1976))
 $k_1 \times k_2 \times k_3$ points + simple cubic, FCC, BCC ...
- Other techniques ... (tetrahedron method)

BZ integration : Monkhorst-Pack grid

- Uniformly spaced grid of $n_{k1} \times n_{k2} \times n_{k3}$ points in the first Brillouin Zone [Monkhorst & Pack, Phys. Rev. B 13, 5188 (1976)]



$$n_{k1} = n_{k2} = 3$$

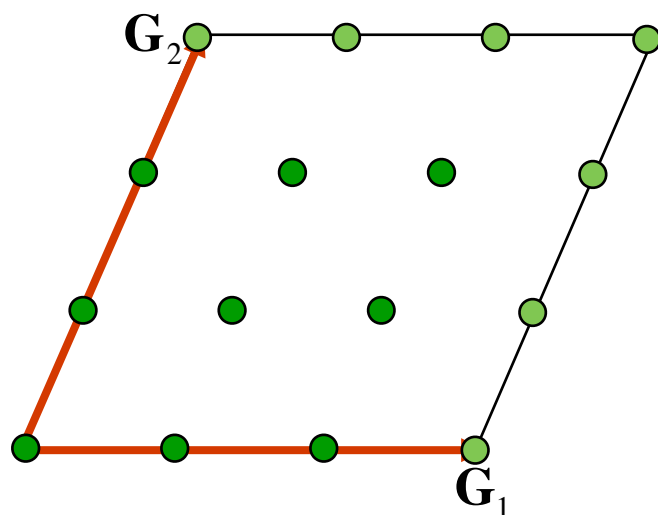


$$n_{k1} = n_{k2} = 4$$

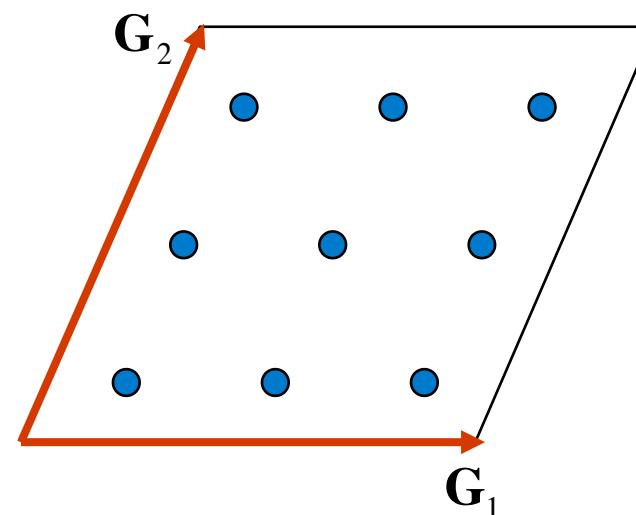
ngkpt nk1 nk2 nk3

Unshifted and shifted grids

- k-points grid can be chosen to be shifted : not centered at Γ .
- Advantage : comparable accuracy can be obtained with fewer k-points in IBZ (especially for highly symmetric cases)



$n_{k1} = n_{k2} = 3$
unshifted

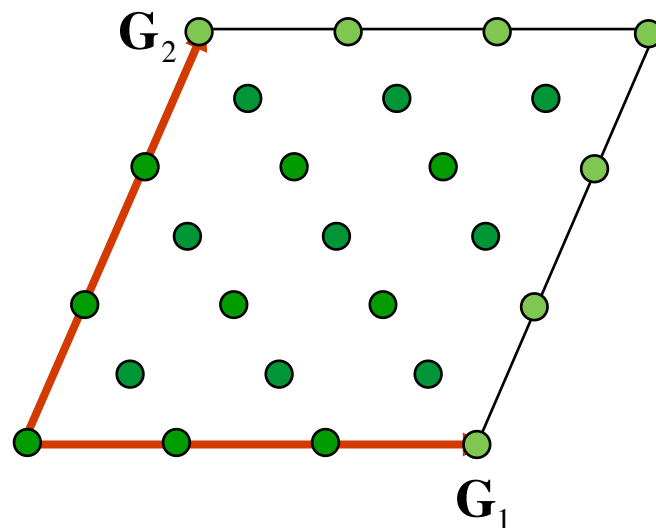


$n_{k1} = n_{k2} = 3$
shifted by $(1/2, 1/2)$

ngkpt nk1 nk2 nk3
shiftk sk1 sk2 sk3 (default: 0.5 0.5 0.5)

Combining grids with various shifts

- k-points grid with various shifts can also be combined.



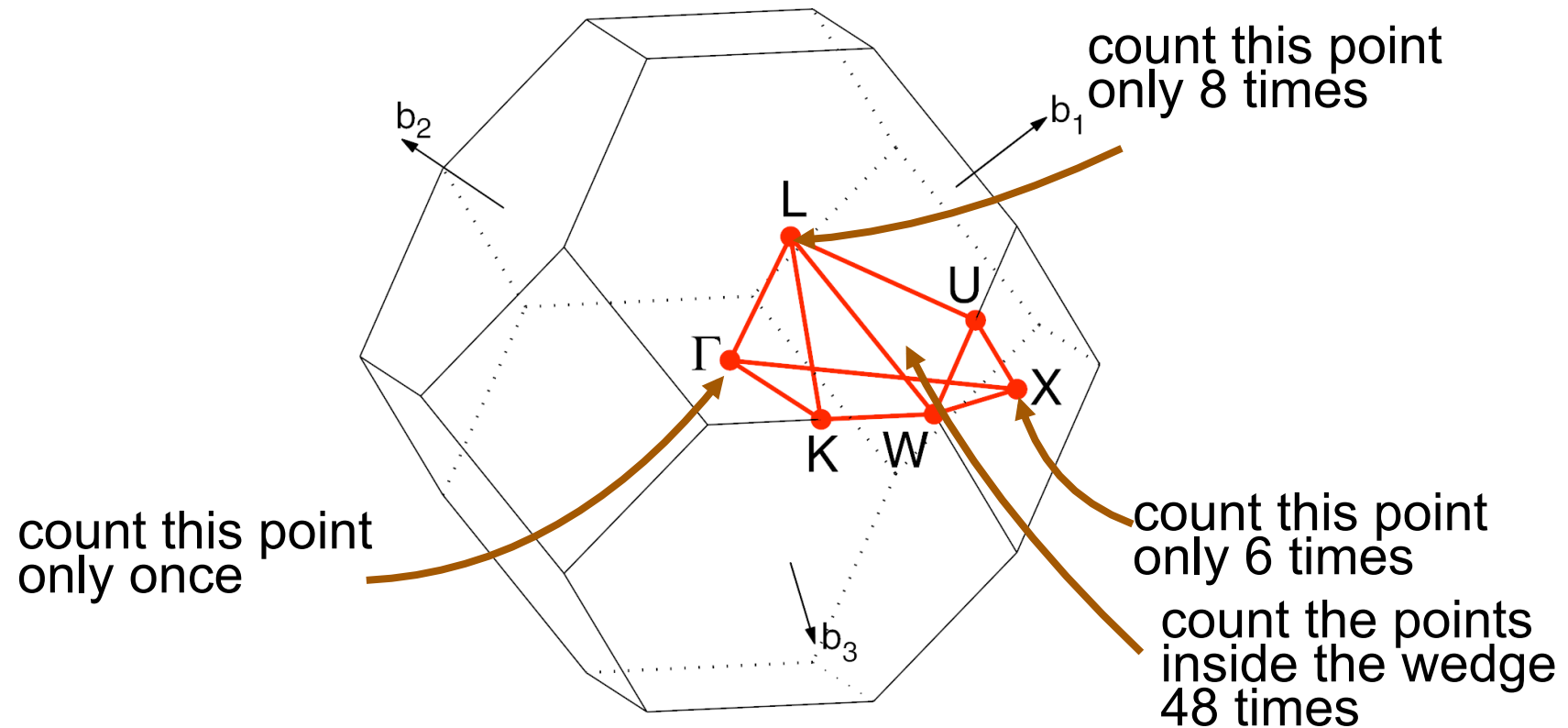
combining unshifted
and shifted by $(1/2, 1/2)$
for $n_{k1} = n_{k2} = 3$

ngkpt	nk1	nk2	nk3
nshiftk	nsk		
shiftk	sk1(1)	sk2(1)	sk3(1)
	sk1(2)	sk2(2)	sk3(2)

	sk1(nsk)	sk2(nsk)	sk3(nsk)

Irreducible wedge

- Using symmetries to avoid summing entire BZ :
- Restrict the sum to the Irreducible Brillouin zone (IBZ) provided that weights are adapted.

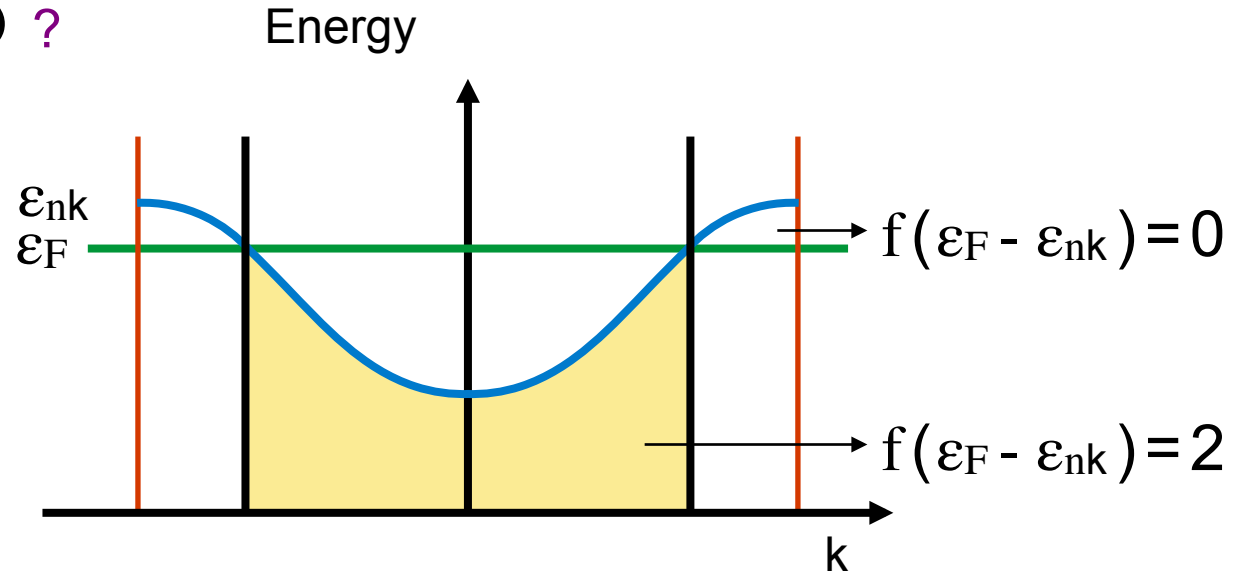


Treatment of metals (I)

Behaviour of $f(\varepsilon_F - \varepsilon_{nk})$?

Discontinuity of integrand at Fermi level

Smearing technique



First trial : generalisation of DFT to finite temperature

$$f(\varepsilon_{nk}) = \frac{1}{1 + e^{(\varepsilon_{nk} - \varepsilon_F)/kT}}$$

f goes from 0 to 2 in an energy range $\sigma = k_B T$

$$E(T) \equiv E(T=0) + \alpha T^2 + \dots$$

$$F(T) = E - TS$$

occopt 3
tsmear σ

Problem : T needed to recover the same convergence as for semiconductors is very high ($\gg 2000$ K)

Treatment of metals (II)

Better technique : obtain $E(\sigma = 0)$ from total energy expression $E(\sigma)$ with **modified occupation numbers**, and σ **similar** to a temperature

$$E(\sigma) = E(\sigma = 0) + \alpha\sigma^2 + O(\sigma^3) \quad \text{with } \alpha \text{ small}$$

$$\text{or } E(\sigma) = E(\sigma = 0) + \alpha\sigma^n + O(\sigma^{n+1}) \quad \text{with } n > 2$$

$$f_{\mathbf{n}\mathbf{k}}(\epsilon_{\mathbf{n}\mathbf{k}}) = s \cdot \int_{t=\frac{\epsilon_{\mathbf{n}\mathbf{k}} - \epsilon_F}{\sigma}}^{\infty} \tilde{\delta}(t) dt \quad [\text{with } \int_{-\infty}^{\infty} \tilde{\delta}(t) dt = 1]$$

Spin factor \nearrow

Gaussian smearing $\tilde{\delta}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \Rightarrow \alpha \text{ small} \quad \text{occupt 7}$

Gauss - Hermite smearing $\tilde{\delta}(x) = \frac{1}{\sqrt{\pi}} \left(\frac{3}{2} - x^2\right) e^{-x^2}$

$\Rightarrow n = 4$ but occupations can be negative

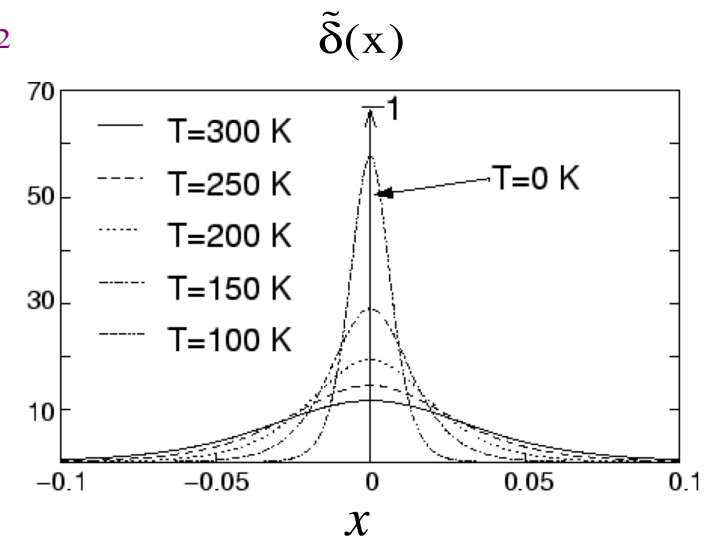
...

'Cold Smearing'

(Marzari et al, Phys. Rev. Lett. 82, 3296 (1999))

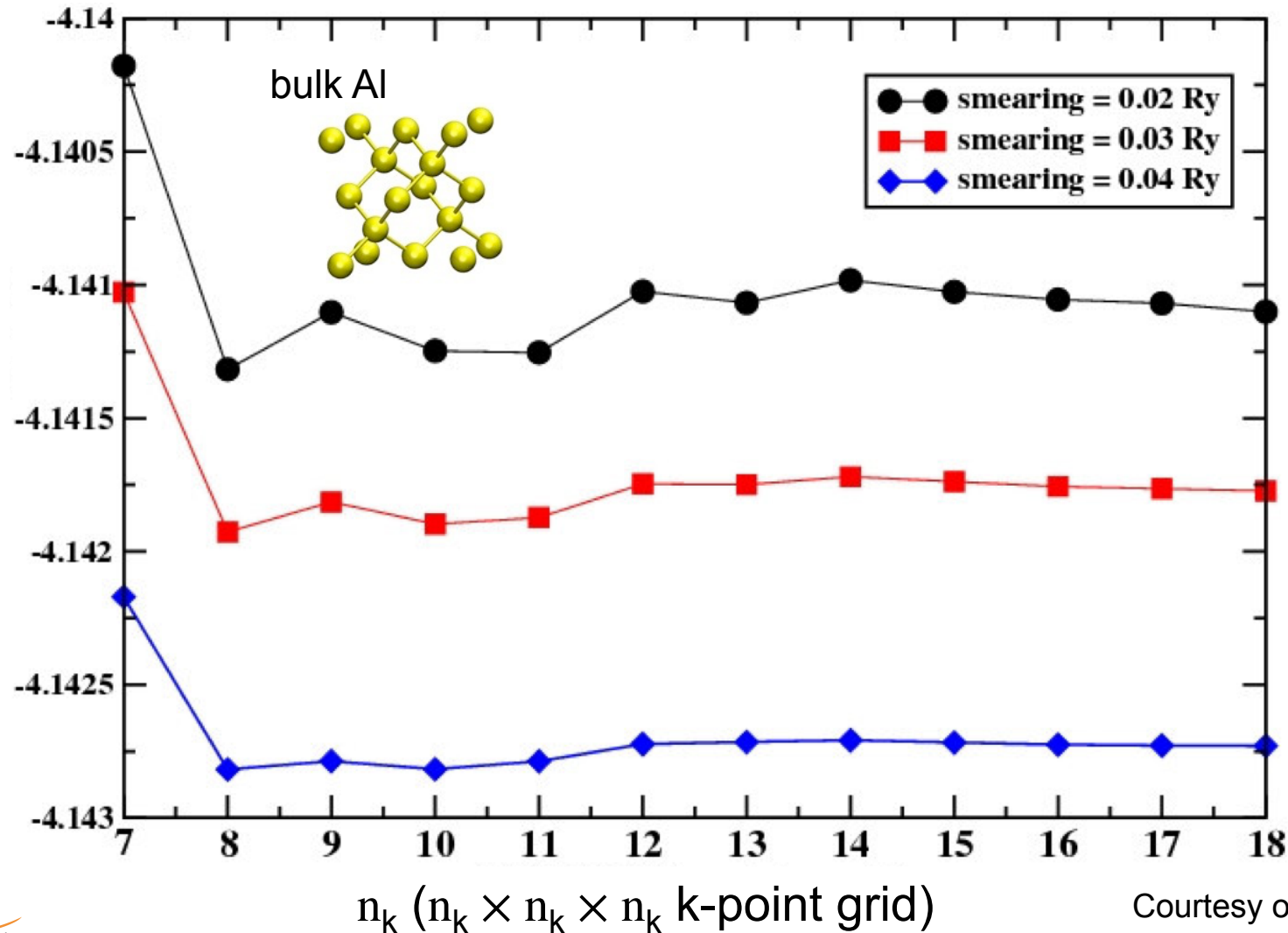
occupt 4/5

$\Rightarrow n = 3$ with positive occupations



Convergence wrt k-points and smearing

Total energy (Ry)



Courtesy of S. Narasimhan

How many k points ? Smearing width ?

Rule of thumb ! Goal : lattice parameter converged better than 0.5 %

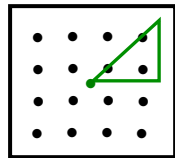
Semiconductors - Insulators # $\mathbf{k} \times N_{atoms}$ 50 ... 500

Metals # $\mathbf{k} \times N_{atoms}$ 1000 ... 2000



Use symmetries \Rightarrow integration in the **irreducible** Brillouin zone

2D Example



grid $4 \times 4 = 16$



3 points in the irreducible Brillouin Zone

Smearing : depends on the density of electronic states (DOS) at the Fermi level

s-p Metal (Al, Na ...) ~ 0.04 Ha

d Metal (Cu, Ag...) ~ 0.01 Ha



magnetism needs small σ

Computing the forces

Computing the forces (I)

Born - Oppenheimer approx. \Rightarrow find electronic ground state in potential created by nuclei.

A starting configuration of nuclei $\{\mathbf{R}_\mathbf{k}\}$ is usually NOT in equilibrium geometry.

$$\mathbf{F}_{\mathbf{k},\alpha} = - \left. \frac{\partial E}{\partial R_{\mathbf{k},\alpha}} \right|_{\{\bar{\mathbf{R}}_\mathbf{k}\}} \quad (\text{principle of virtual works})$$

Forces can be computed by finite differences.

Better approach : compute the response to a perturbation

\Rightarrow What is the energy change ?

$$\{\mathbf{R}_{\mathbf{k},\alpha}\} \rightarrow \{\mathbf{R}_{\mathbf{k},\alpha} + \lambda \delta \mathbf{R}_{\mathbf{k},\alpha}\}$$

Small parameter

Computing the forces (II)

To simplify, let's compute the derivative of an electronic eigenvalue

Perturbation theory : Hellmann - Feynman theorem

$$\frac{d\varepsilon_n}{d\lambda} = \left\langle \psi_n^{(0)} \left| \frac{d\hat{H}}{d\lambda} \right| \psi_n^{(0)} \right\rangle$$

$\frac{d\psi_n}{d\lambda}$ not needed !

Application to the derivative with respect to an atomic displacement :

$$\hat{H} = \hat{T} + \hat{V}_{ext} \{ \hat{\mathbf{R}} \} \Rightarrow \frac{\partial \hat{H}}{\partial \mathbf{R}_{\kappa, \alpha}} = \frac{\partial \hat{V}_{ext}}{\partial \mathbf{R}_{\kappa, \alpha}}$$
$$\frac{\partial \varepsilon_n}{\partial \mathbf{R}_{\kappa, \alpha}} = \left\langle \psi_n \left| \frac{\partial \hat{H}}{\partial \mathbf{R}_{\kappa, \alpha}} \right| \psi_n \right\rangle = \int n(\mathbf{r}) \frac{\partial \hat{V}_{ext}(\mathbf{r})}{\partial \mathbf{R}_{\kappa, \alpha}} d\mathbf{r}$$

Computing the forces (III)

Generalisation to density functional theory

Reminder : $E[\psi] = \sum_n \langle \psi_n | \hat{T} | \psi_n \rangle + \int n(\mathbf{r}) V_{\text{ext}}(\mathbf{r}) d\mathbf{r} + E_{\text{Hxc}}[n]$

If change of atomic positions ...

$$V_{\text{ext}}(\vec{r}) = \sum_{k'} -\frac{Z_{k'}}{|\vec{r} - \vec{R}_{k'}|} \quad (\text{can be generalized to pseudopotential case})$$

$$\frac{\partial V_{\text{ext}}(\vec{r})}{\partial R_{k,\alpha}} = +\frac{Z_{k'}}{|\vec{r} - \vec{R}_k|^2} \cdot \frac{\partial |\vec{r} - \vec{R}_k|}{\partial R_{k,\alpha}} = -\frac{Z_{k'}}{|\vec{r} - \vec{R}_k|^3} \cdot (\vec{r} - \vec{R}_k)_\alpha$$

$$\boxed{\frac{\partial E}{\partial R_{k,\alpha}} = \int n(\mathbf{r}') \frac{\partial V_{\text{ext}}(\mathbf{r}')}{\partial R_{k,\alpha}} d\mathbf{r}' = - \int \frac{n(\mathbf{r}')}{|\vec{r}' - \vec{R}_k|^3} \cdot (\vec{r}' - \vec{R}_k)_\alpha d\vec{r}'}$$

Forces can be computed directly from the density !

Iterative algorithms

Algorithmics : problems to be solved

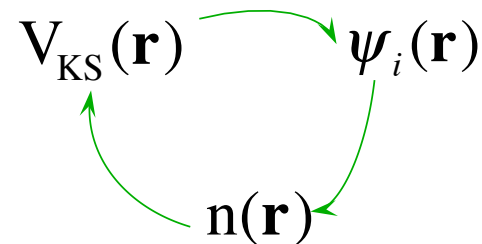
(1) Kohn - Sham equation

$$\underline{\underline{A}} \underline{x}_i = \lambda_i \underline{x}_i$$

$$\left[\begin{array}{c} -\frac{1}{2} \nabla^2 + V_{\text{KS}}(\mathbf{r}) \\ \{\mathbf{G}_j\} \quad \{\mathbf{r}_j\} \end{array} \right] \psi_i(\mathbf{r}) = \varepsilon_i \psi_i(\mathbf{r})$$

Size of the system	[2 atoms...	600 atoms...	+ vacuum ?
Dimension of the vectors \underline{x}_i	300...	100 000...	(if planewaves)
# of (occupied) eigenvectors	4...	1200...	

(2) Self-consistency



(3) Geometry optimization

Find the positions $\{\mathbf{R}_\kappa\}$ of ions such that the forces $\{\mathbf{F}_\kappa\}$ vanish
 [= Minimization of energy]

Current practice : iterative approaches

The 'steepest-descent' algorithm

Forces are gradients of the energy : moving the atoms along gradients is the steepest descent of the energy surface.

=> Iterative algorithm.

Choose a starting geometry, then a parameter λ ,
and iterately update the geometry, following the forces :

$$\mathbf{R}_{K,\alpha}^{(n+1)} = \mathbf{R}_{K,\alpha}^{(n)} + \lambda \mathbf{F}_{K,\alpha}^{(n)}$$

Equivalent to the simple mixing algorithm
of SCF (see later)

Energy+forces around equilib. geometry

Let us denote the equilibrium geometry as $\mathbf{R}_{\kappa,\alpha}^*$

Analysis of forces close to the equilibrium geometry,
at which forces vanish, thanks to a Taylor expansion :

$$F_{\kappa,\alpha}(\mathbf{R}_{\kappa',\alpha'}) = \cancel{F_{\kappa,\alpha}(\mathbf{R}_{\kappa',\alpha'}^*)} + \sum_{\kappa',\alpha'} \left. \frac{\partial F_{\kappa,\alpha}}{\partial \mathbf{R}_{\kappa',\alpha'}} \right|_{\{\mathbf{R}^*\}} (\mathbf{R}_{\kappa',\alpha'} - \mathbf{R}_{\kappa',\alpha'}^*) + O(\mathbf{R}_{\kappa',\alpha'} - \mathbf{R}_{\kappa',\alpha'}^*)^2$$

Moreover,
$$F_{\kappa,\alpha} = -\frac{\partial E^{\text{BO}}}{\partial \mathbf{R}_{\kappa,\alpha}} \quad \frac{\partial F_{\kappa',\alpha'}}{\partial \mathbf{R}_{\kappa,\alpha}} = -\frac{\partial^2 E^{\text{BO}}}{\partial \mathbf{R}_{\kappa,\alpha} \partial \mathbf{R}_{\kappa',\alpha'}}$$

Vector and matrix notation

$$\begin{array}{lll} \mathbf{R}_{\kappa,\alpha}^* \rightarrow \underline{\mathbf{R}}^* & \mathbf{F}_{\kappa,\alpha} \rightarrow \underline{\mathbf{F}} & \left. \frac{\partial^2 E^{\text{BO}}}{\partial \mathbf{R}_{\kappa,\alpha} \partial \mathbf{R}_{\kappa',\alpha'}} \right|_{\{\mathbf{R}_{\kappa,\alpha}^*\}} \rightarrow \underline{\underline{\mathbf{H}}} \\ \mathbf{R}_{\kappa,\alpha} \rightarrow \underline{\mathbf{R}} & & \text{(the Hessian)} \end{array}$$

Steepest-descent : analysis (I)

$$\underline{R}_{K,\alpha}^{(n+1)} = \underline{R}_{K,\alpha}^{(n)} + \lambda \underline{F}_{K,\alpha}^{(n)}$$

Analysis of this algorithm, in the linear regime :

$$\underline{F}(\underline{R}) = \underline{F}(\underline{R}^*) - \underline{H}(\underline{R} - \underline{R}^*) + O(\underline{R} - \underline{R}^*)^2$$

$$\underline{R}^{(n+1)} = \underline{R}^{(n)} + \lambda \underline{F}^{(n)} \longrightarrow (\underline{R}^{(n+1)} - \underline{R}^*) = (\underline{R}^{(n)} - \underline{R}^*) - \lambda \underline{H}(\underline{R}^{(n)} - \underline{R}^*)$$

$$(\underline{R}^{(n+1)} - \underline{R}^*) = (\underline{1} - \lambda \underline{H})(\underline{R}^{(n)} - \underline{R}^*)$$

For convergence of the iterative procedure, the "distance" between trial geometry and equilibrium geometry must decrease.

- 1) Can we predict conditions for convergence ?
- 2) Can we make convergence faster ?

Need to understand the action of the matrix (or operator)

$$\underline{1} - \lambda \underline{H}$$

Steepest-descent : analysis (II)

What are the eigenvectors and eigenvalues of $\underline{\underline{H}}$?

$\underline{\underline{H}}$ symmetric,
positive definite matrix

$$\left(= \frac{\partial^2 E^{\text{BO}}}{\partial \mathbf{R}_{\kappa, \alpha} \partial \mathbf{R}_{\kappa', \alpha'}} \bigg|_{\{\mathbf{R}_{\kappa, \alpha}^*\}} \right)$$

$\underline{\underline{H}} \underline{\underline{f}}_i = h_i \underline{\underline{f}}_i$ where $\{\underline{\underline{f}}_i\}$ form a complete, orthonormal, basis set

Discrepancy decomposed as $(\underline{\underline{R}}^{(n)} - \underline{\underline{R}}^*) = \sum c_i^{(n)} \underline{\underline{f}}_i$

and
$$(\underline{\underline{R}}^{(n+1)} - \underline{\underline{R}}^*) = (\underline{\underline{1}} - \lambda \underline{\underline{H}}) \sum_i c_i^{(n)} \underline{\underline{f}}_i = \sum_i c_i^{(n)} (1 - \lambda h_i) \underline{\underline{f}}_i$$

The coefficient of $\underline{\underline{f}}_i$ is multiplied by $1 - \lambda h_i$

Iteratively :

$$(\underline{\underline{R}}^{(n)} - \underline{\underline{R}}^*) = \sum_i c_i^{(0)} (1 - \lambda h_i)^{(n)} \underline{\underline{f}}_i$$

Steepest-descent : analysis (III)

$$\left(\underline{\mathbf{R}}^{(n)} - \underline{\mathbf{R}}^*\right) = \sum_i c_i^{(0)} (1 - \lambda h_i)^{(n)} \underline{\mathbf{f}}_i$$

The size of the discrepancy decreases if $|1 - \lambda h_i| < 1$

Is it possible to have $|1 - \lambda h_i| < 1$, for all eigenvalues ?

H positive definite \Rightarrow all h_i are positive

Yes ! If λ positive, sufficiently small ...

Steepest-descent : analysis (IV)

$$\left(\underline{\mathbf{R}}^{(n)} - \underline{\mathbf{R}}^*\right) = \sum_i c_i^{(0)} (1 - \lambda h_i)^{(n)} \underline{\mathbf{f}}_i$$

How to determine the optimal value of λ ?

The maximum of all $|1 - \lambda h_i|$ should be as small as possible.

At the optimal value of λ , what will be the convergence rate ?

(= by which factor is reduced the worst component of $\left(\underline{\mathbf{R}}^{(n)} - \underline{\mathbf{R}}^*\right)$?)

As an exercise : suppose $\left. \begin{array}{l} h_1 = 0.2 \\ h_2 = 1.0 \\ h_3 = 5.0 \end{array} \right\} \Rightarrow \text{what is the best value of } \lambda \text{ ?}$
+ what is the convergence rate ?

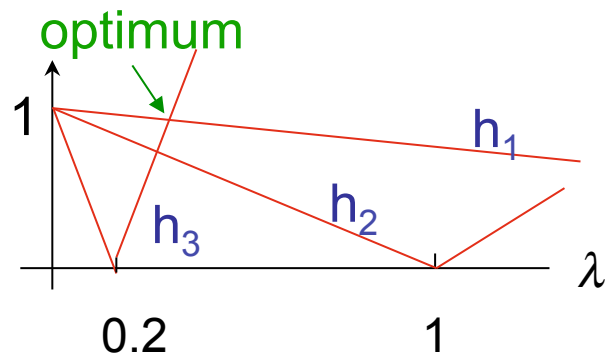
Hint : draw the three functions $|1 - \lambda h_i|$ as a function of λ . Then, find the location of λ where the largest of the three curves is the smallest.
Find the coordinates of this point.

Steepest-descent : analysis (V)

Minimise the maximum of $|1 - \lambda h_i|$

$$\begin{array}{l} h_1 = 0.2 \\ h_2 = 1.0 \\ h_3 = 5.0 \end{array} \quad \begin{array}{l} |1 - \lambda \cdot 0.2| \\ |1 - \lambda \cdot 1| \\ |1 - \lambda \cdot 5| \end{array}$$

$$\begin{array}{l} \text{optimum} \Rightarrow \lambda = 5 \\ \text{optimum} \Rightarrow \lambda = 1 \\ \text{optimum} \Rightarrow \lambda = 0.2 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{optimum} \Rightarrow \lambda = 5 \\ \text{optimum} \Rightarrow \lambda = 1 \\ \text{optimum} \Rightarrow \lambda = 0.2 \end{array}} \right\} ?$$



$$\mu = |1 - \underbrace{\lambda \cdot 0.2}_{\text{positive}}| = |1 - \underbrace{\lambda \cdot 5}_{\text{negative}}|$$

$$1 - \lambda \cdot 0.2 = -(1 - \lambda \cdot 5)$$

$$2 - \lambda (0.2 + 5) = 0 \Rightarrow \lambda = 2/5.2$$

$$\mu = 1 - 2 \cdot (0.2 / 5.2)$$

Only ~ 8% decrease of the error, per iteration ! Hundreds of iterations will be needed to reach a reduction of the error by 1000 or more.

Note : the second eigenvalue does not play any role.

The convergence is limited by the extremal eigenvalues : if the parameter is too large, the smallest eigenvalue will cause divergence, but for that small parameter, the largest eigenvalue lead to slow decrease of the error...

The condition number

In general, $\lambda_{\text{opt}} = 2 / (h_{\text{min}} + h_{\text{max}})$
 $\mu_{\text{opt}} = 2 / [1 + (h_{\text{max}}/h_{\text{min}})] - 1 = [(h_{\text{max}}/h_{\text{min}}) - 1] / [(h_{\text{max}}/h_{\text{min}}) + 1]$

Perfect if $h_{\text{max}} = h_{\text{min}}$. Bad if $h_{\text{max}} \gg h_{\text{min}}$.

$h_{\text{max}}/h_{\text{min}}$ called the "condition" number. A problem is "ill-conditioned" if the condition number is large. It does **not** depend on the intermediate eigenvalues.

Suppose we start from a configuration with forces on the order of 1 Ha/Bohr, and we want to reach the target $1e-4$ Ha/Bohr. The mixing parameter is optimal.

How many iterations are needed ?

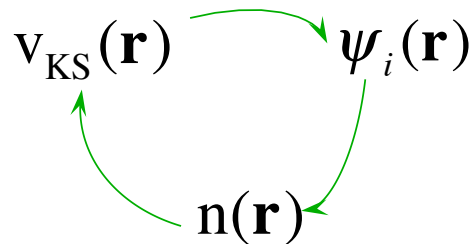
For a generic decrease factor Δ , with "n" the number of iterations.

$$\|\underline{F}^{(n)}\| \approx \left(\frac{h_{\text{max}}/h_{\text{min}} - 1}{h_{\text{max}}/h_{\text{min}} + 1} \right)^n \|\underline{F}^{(0)}\| \quad \Delta \approx \left(\frac{h_{\text{max}}/h_{\text{min}} - 1}{h_{\text{max}}/h_{\text{min}} + 1} \right)^n$$

$$n \approx \left[\ln \left(\frac{h_{\text{max}}/h_{\text{min}} + 1}{h_{\text{max}}/h_{\text{min}} - 1} \right) \right]^{-1} \ln \Delta \approx 0.5 (h_{\text{max}}/h_{\text{min}}) \ln \frac{1}{\Delta}$$

(The latter approximate equality supposes a large condition number)

Analysis of self-consistency



Natural iterative methodology (KS : in => out) :

$$v_{\text{in}}(\mathbf{r}) \rightarrow \psi_i(\mathbf{r}) \rightarrow n(\mathbf{r}) \rightarrow v_{\text{out}}(\mathbf{r})$$

Which quantity plays the role of a force, that should vanish at the solution ?

The difference

$$v_{\text{out}}(\mathbf{r}) - v_{\text{in}}(\mathbf{r}) \quad (\text{generic name : a "residual"})$$

Simple mixing algorithm
(\approx steepest - descent)

$$\underline{v}_{\text{in}}^{(n+1)} = \underline{v}_{\text{in}}^{(n)} + \lambda \left(\underline{v}_{\text{out}}^{(n)} - \underline{v}_{\text{in}}^{(n)} \right)$$

$$\text{Analysis ... } \underline{v}_{\text{out}} \left[\underline{v}_{\text{in}} \right] = \underline{v}_{\text{out}} \left[\underline{v}^* \right] + \frac{\delta \underline{v}_{\text{out}}}{\delta \underline{v}_{\text{in}}} \left(\underline{v}_{\text{in}} - \underline{v}^* \right)$$

\searrow
 $\underline{\underline{H}}$

Like the steepest-descent algorithm, this leads to the requirement to minimize $|1 - \lambda h_i|$ where h_i are eigenvalues of

$$\frac{\delta \underline{v}_{\text{out}}}{\delta \underline{v}_{\text{in}}}$$

Modify the condition number (II)

$$\underline{\underline{R}}^{(n+1)} = \underline{\underline{R}}^{(n)} + \lambda \left(\underline{\underline{H}}^{-1} \right)_{\text{approx}} \underline{\underline{F}}^{(n)}$$

$$\underline{\underline{F}}(\underline{\underline{R}}) = -\underline{\underline{H}}(\underline{\underline{R}} - \underline{\underline{R}}^*) \implies (\underline{\underline{R}}^{(n+1)} - \underline{\underline{R}}^*) = \left(1 - \lambda \left(\underline{\underline{H}}^{-1} \right)_{\text{approx}} \underline{\underline{H}} \right) (\underline{\underline{R}}^{(n)} - \underline{\underline{R}}^*)$$

- Notes :
- 1) If approximate inverse Hessian perfect, optimal geometry is reached in one step, with $\lambda = 1$.
Steepest-descent NOT the best direction.
 - 2) Non-linear effects not taken into account. For geometry optimization, might be quite large. Even with perfect hessian, need 5-6 steps to optimize a water molecule.
 - 3) Approximating inverse hessian by a multiple of the unit matrix is equivalent to changing the λ value.
 - 4) Eigenvalues and eigenvectors of $\left(\underline{\underline{H}}^{-1} \right)_{\text{approx}} \underline{\underline{H}}$ govern the convergence : the condition number can be changed.
 $\left(\underline{\underline{H}}^{-1} \right)_{\text{approx}}$ often called a "**pre-conditioner**".
 - 5) Generalisation to other optimization problems is trivial.
(The Hessian is referred to as the Jacobian if it is not symmetric.)

Modify the condition number (III)

Approximate Hessian can be generated on a case-by-case basis.

Selfconsistent determination of the Kohn-Sham potential :

Jacobian = dielectric matrix.

Lowest eigenvalue **close to 1**.

Largest eigenvalue :

= **1.5 ... 2.5** for small close-shell molecules, and small unit cell solids

(Simple mixing will sometimes converge with parameter set to 1 !)

= the **macroscopic dielectric constant** (e.g. 12 for silicon),

for larger close-shell molecules and large unit cell insulators,

= **diverge** for large-unit cell metals, or open-shell molecules !

Model dielectric matrices known for rather homogeneous systems.

Knowledge of approx. macroscopic dielectric constant

=> efficient preconditioner

Work in progress for inhomogeneous systems

(e.g. metals/vacuum systems).

Advanced algorithms : using the history

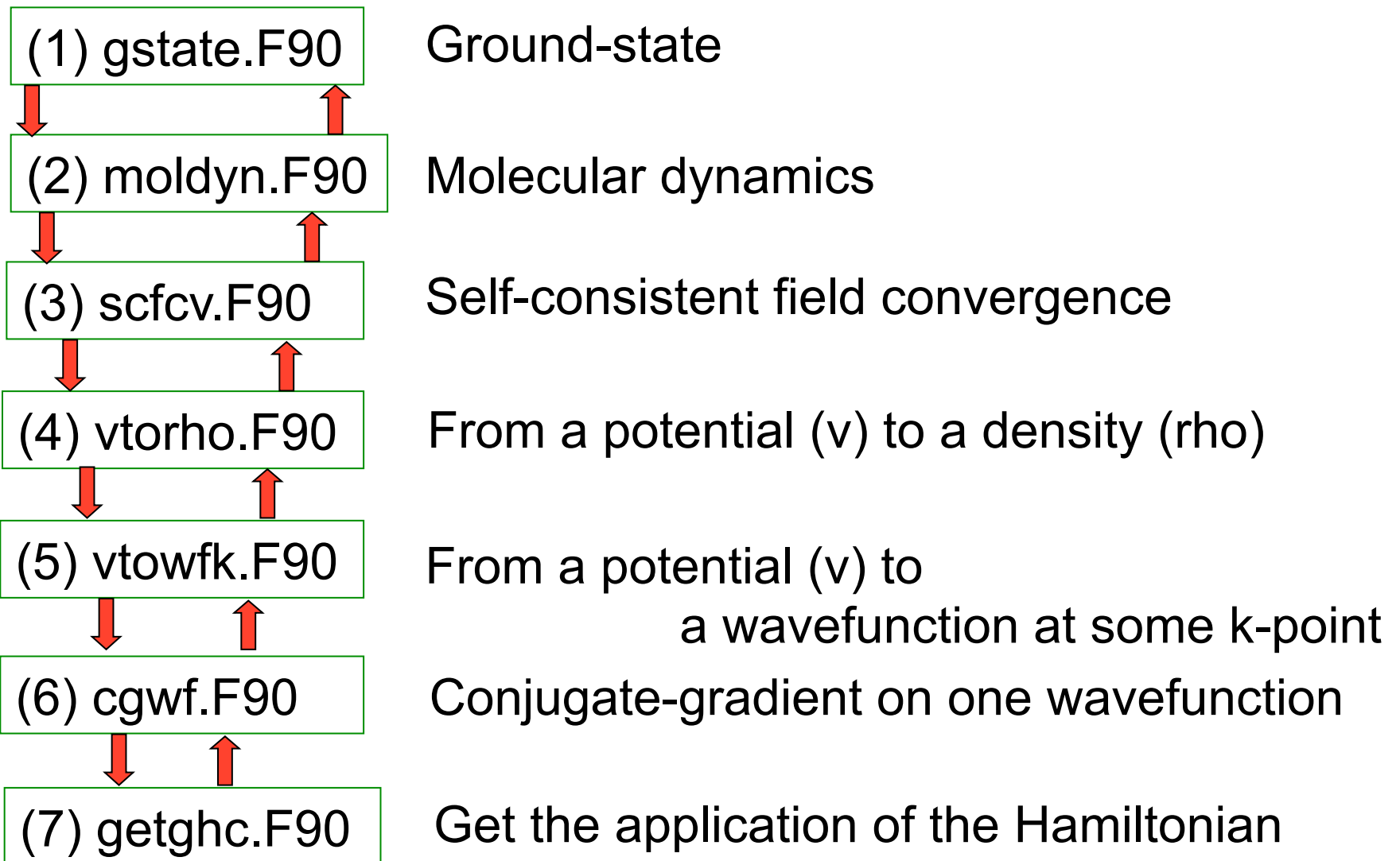
Instead of using only previously computed forces,
take into account past forces for past positions

Large class of methods :

- Broyden (quasi-Newton-type),
- Davidson,
- conjugate gradients,
- Lanczos ...

Approximate Hessian can be combined with usage of history

Stages in the main processing unit



In practice ...

(1) Kohn - Sham equation
$$\left[-\frac{1}{2} \nabla^2 + V_{\text{KS}}(\mathbf{r}) \right] \psi_i(\mathbf{r}) = \varepsilon_i \psi_i(\mathbf{r})$$

Details are usually hidden to the user

nline

Note that scaling with size of system is quadratic or even cubic

(2) Self-consistency

Target tolerance

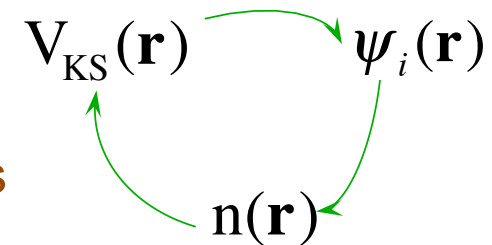
toldfe, toldff, tolrff, tolvrs

+ Maximal number of loops

nstep

Preconditioner

diemac, diemix, ...



(3) Geometry optimization / molecular dynamics

Find the positions $\{\mathbf{R}_k\}$ of ions such that the forces $\{\mathbf{F}_k\}$ vanish

Target tolerance

tolmxf

+ Maximal number of loops

ntime

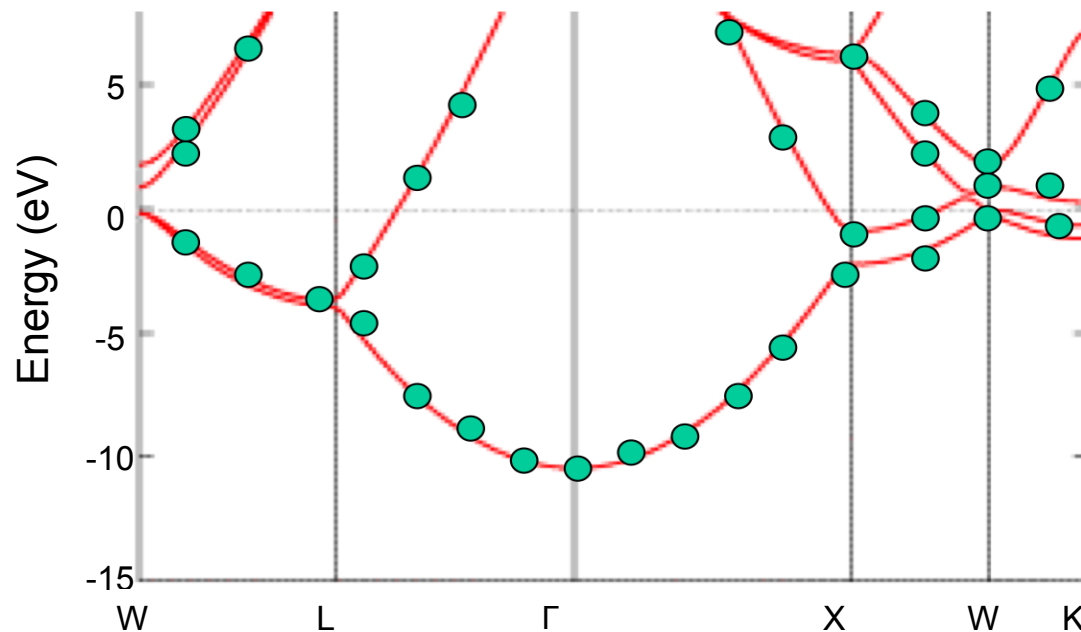
Algorithm

ionmov

Computing band structure and density of states

Non-self consistent calculations

- Once the density has been determined self-consistently, it is possible to compute the eigenenergies/eigenfunctions rapidly for a large number of wavevectors, at fixed KS potential
- Band structure :
non self-consistent calculation (`iscf -2`)
k-points along high-symmetry directions (`kptopt<0;kptbounds;ndivk`).



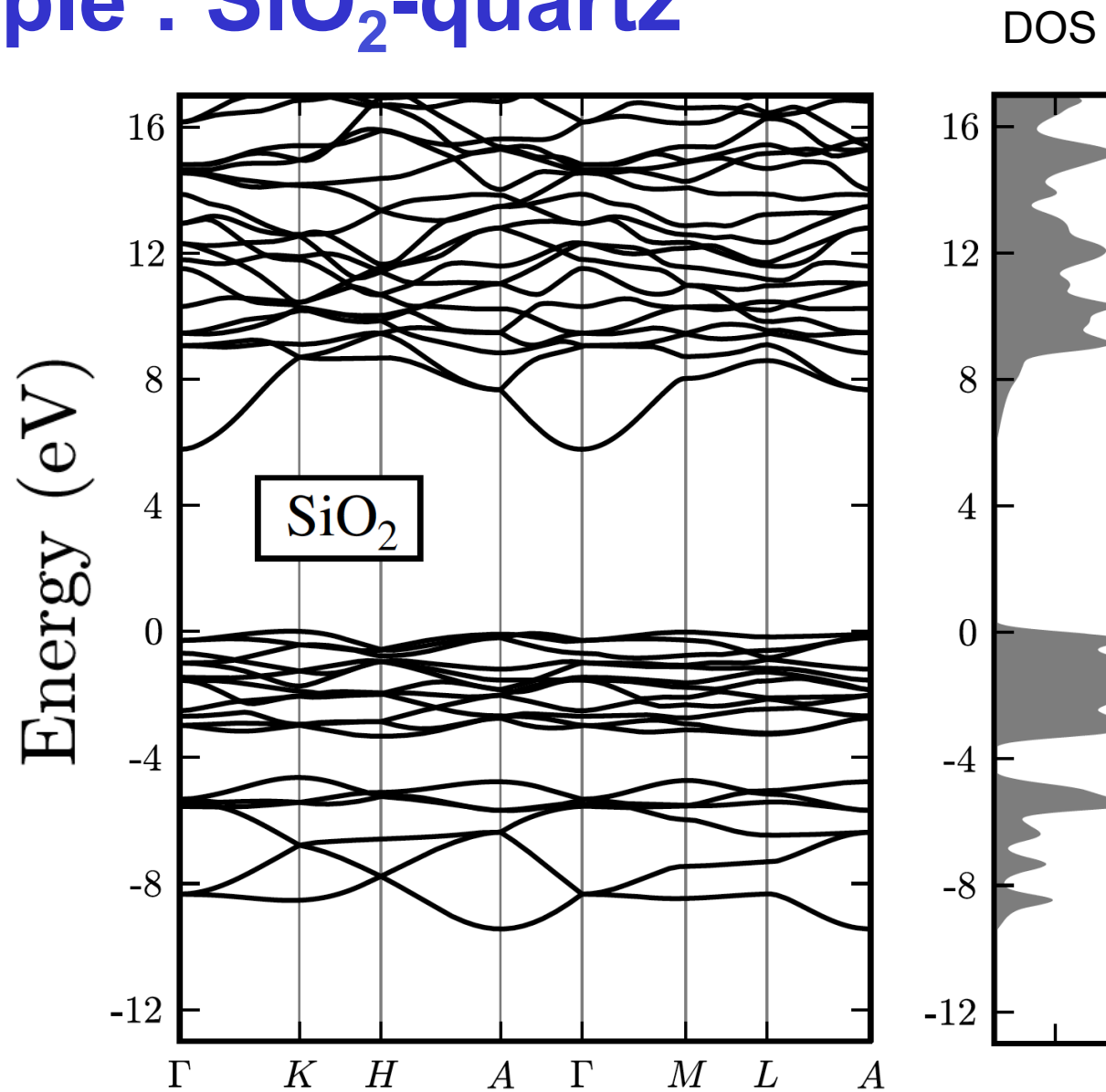
Density of states

- Density of states (DOS) defined as number of states available in energy range E to $E+dE$:

$$g(E) = \sum_n \frac{1}{\Omega_{0\mathbf{k}}} \int_{\Omega_{0\mathbf{k}}} \delta(E - \varepsilon_{n\mathbf{k}}) d\mathbf{k}$$

- Recipe : determine $\varepsilon_{n\mathbf{k}}$ on a grid of k-points in the BZ using a non self-consistent procedure (`iscf -3`).
- The δ -function is approximated by smeared-out function (typically a Gaussian) with a width σ (`prtdos 1`).
- Very fine grid of k-points needed to get sharp features accurately. For faster convergence, use tetrahedron method to interpolate between k-points (`prtdos 2`).

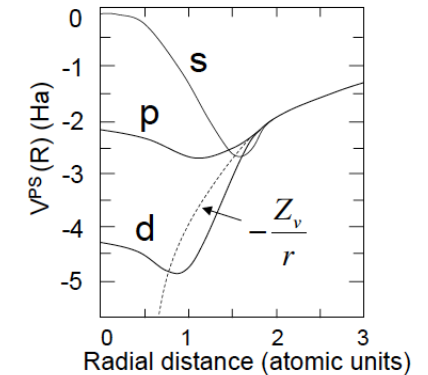
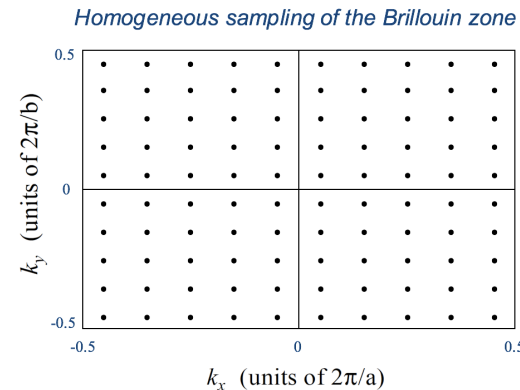
Example : SiO₂-quartz



Summary

- Plane waves basis set
- Brillouin zone integration
- PW need pseudopotentials
- Easy computation of forces

$$\psi_{\mathbf{k}}(\mathbf{r}) = (N\Omega_0)^{-1/2} \sum_{\mathbf{G}} u_{\mathbf{k}}(\mathbf{G}) e^{i(\mathbf{k}+\mathbf{G})\mathbf{r}}$$



=> relaxation of geometry, or molecular dynamics

